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# Compromise solutions based on bankruptcy

Marieke Quant<sup>1,2</sup>      Peter Borm<sup>2</sup>      Ruud Hendrickx<sup>2</sup>  
Peter Zwikker

## Abstract

A new family of compromise solutions is introduced for the class of compromise admissible games. These solutions extend bankruptcy rules. It is shown that the compromise extension of the run-to-the-bank rule coincides with the average of the extreme points of the core cover (taking multiplicities into account) and that this solution is characterised by means of a recursive formula.

**Keywords:** Bankruptcy, run to the bank rule, compromise admissible games.

**JEL Classification Number:** C71.

## 1 Introduction

The model of bankruptcy situations as introduced by O'Neill (1982) is a general framework for various kinds of simple allocation problems. In a bankruptcy problem, there is an estate to be divided and each player has a single claim on the estate. The total of the claims is larger than the estate available, so one has to find criteria on the basis of which the estate is to be divided. In this context, many rules have been proposed to come to a fair allocation of the estate. For a recent overview of such rules, the reader is referred to Thomson (2003).

A bankruptcy situation can be seen as the most basic form of an allocation problem. As a consequence, many bankruptcy rules have a straightforward interpretation and appropriate properties of such rules are easily formulated. In a transferable utility game, the allocation problem is of a

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<sup>1</sup>Corresponding author. Email: [quant@uvt.nl](mailto:quant@uvt.nl).

<sup>2</sup>Department of Econometrics & OR and CentER, Tilburg University, P.O. Box 90153, 5000 LE, Tilburg, The Netherlands.

more complicated nature: instead of each player having a single claim, each coalition of players has a worth which has to be taken into account. Our aim is to extend bankruptcy rules to the class of transferable utility games in such a way that both the interpretation and the appealing properties are maintained.

In this paper, we provide such an extension to the class of compromise admissible (or quasi-balanced) games (cf. Tijs and Lipperts (1982)). Quant et al. (2003) study the compromise extension of the Talmud rule for the class of games for which the core coincides with the core cover. They prove that for this class of games the compromise extension of the Talmud rule coincides with the nucleolus. González-Díaz et al. (2003) introduce the compromise extension of the adjusted proportional rule and show that this solution coincides with the barycentre of the edges of the core cover. In the current paper, we look at the problem of extending bankruptcy rules from a more general viewpoint.

An important concept in the bankruptcy literature is duality (cf. Aumann and Maschler (1985)). We use this notion to define for each rule a dual compromise extension and show that this solution coincides with the compromise extension of the dual rule.

We pay particular attention to one specific solution: the compromise extension of the run-to-the-bank-rule. We show that this solution is the average of the extreme points of the core cover (taking multiplicities into account) and characterise it by a recursive formula.

This paper is organised as follows. In section 2 we present some basic definitions concerning transferable utility games and bankruptcy situations. In section 3, we define the concept of compromise extension and analyse the dual extension. Section 4 deals with the run-to-the-bank rule and shows that this solution is the average of the extreme points of the core cover. Finally, in section 5 we provide a recursive formula for the extension of the run-to-the-bank rule.

## 2 Preliminaries

For two sets  $A, B$  we denote  $A \subset B$  if for all  $i \in A$  we have  $i \in B$ . For finite  $A$  and two vectors  $x, y \in \mathbb{R}^A$  we write  $x \leq y$  if  $x_i \leq y_i$  for all  $i \in A$ .

A *transferable utility game* (in short *TU game*) is a pair  $(N, v)$ , where  $N$  denotes a finite set of players and  $v : 2^N \rightarrow \mathbb{R}$  is a function assigning to each coalition  $S \in 2^N$  a worth  $v(S)$ . By convention  $v(\emptyset) = 0$ . The set of

all TU games with player set  $N$  is denoted by  $TU^N$ . Where no confusion arises, we write  $v$  rather than  $(N, v)$ .

Let  $v \in TU^N$ . The *utopia demand* of a player  $i \in N$ ,  $M_i(v)$  is defined by

$$M_i(v) = v(N) - v(N \setminus \{i\}).$$

The *minimum right* of a player  $i \in N$ ,  $m_i(v)$ , is the minimum value this player can achieve by satisfying all other players in a coalition by giving them their utopia demands:

$$m_i(v) = \max_{S: i \in S} \left\{ v(S) - \sum_{j \in S \setminus \{i\}} M_j(v) \right\}.$$

The *core cover* of a game  $v \in TU^N$ ,  $CC(v)$ , consists of all efficient allocation vectors, such that no player receives more than his utopia payoff or less than his minimum right:

$$CC(v) = \left\{ x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N), m_i(v) \leq x_i \leq M_i(v) \right\}.$$

A game is called *compromise admissible* if it has a nonempty core cover. The class of all compromise admissible games with player set  $N$  is denoted by  $CA^N$ . From the definition of the core cover it immediately follows that  $v \in CA^N$  if and only if  $m_i(v) \leq M_i(v)$  and  $\sum_{i \in N} m_i(v) \leq v(N) \leq \sum_{i \in N} M_i(v)$ .

The core cover is a polytope with at most  $|N|!$  extreme points. These so-called *larginal vectors* are introduced in Quant et al. (2003) and have been extensively studied in González-Díaz et al. (2003).

An order on  $N$  is a bijective function  $\sigma : \{1, \dots, |N|\} \rightarrow N$ . The player at position  $k$  in the order  $\sigma$  is denoted by  $\sigma(k)$ . The set of all orders on  $N$  is denoted by  $\Pi(N)$ . For  $\sigma \in \Pi(N)$ , the *larginal*  $\ell^\sigma(v)$  is the efficient payoff vector giving the first players in  $\sigma$  their utopia demands as long as it is still possible to satisfy the remaining players with their minimum rights.

Let  $v \in CA^N$  and  $\sigma \in \Pi(N)$ . The *larginal vector*  $\ell^\sigma(v)$  is defined by

$$\ell_{\sigma(k)}^\sigma(v) = \begin{cases} M_{\sigma(k)}(v) & \text{if } \sum_{r=1}^k M_{\sigma(r)}(v) + \sum_{r=k+1}^{|N|} m_{\sigma(r)}(v) \leq v(N), \\ m_{\sigma(k)}(v) & \text{if } \sum_{r=1}^{k-1} M_{\sigma(r)}(v) + \sum_{r=k}^{|N|} m_{\sigma(r)}(v) \geq v(N), \\ v(N) - \sum_{r=1}^{k-1} M_{\sigma(r)}(v) - \sum_{r=k+1}^{|N|} m_{\sigma(r)}(v) & \text{otherwise} \end{cases}$$

for every  $k \in \{1, \dots, |N|\}$ .

It is readily seen that the core cover equals the convex hull of all larginals

$$CC(v) = \text{conv}\{\ell^\sigma(v) \mid \sigma \in \Pi(N)\}.$$

An (one point) solution  $f$  on a subclass  $A \subset TU^N$  is a function  $f : A \rightarrow \mathbb{R}^N$  assigning to each game  $v \in A$  a payoff vector  $f(v) \in \mathbb{R}^N$ . This paper introduces a new type of allocation rule on  $CA^N$  based on bankruptcy situations.

A *bankruptcy situation* is a triple  $(N, E, d)$ , often abbreviated to  $(E, d)$ .  $N$  is a finite set of players,  $E \geq 0$  is the estate which has to be divided among the players and  $d \in \mathbb{R}_+^N$  is a vector of claims, where for  $i \in N$ ,  $d_i$  represents player  $i$ 's claim on the estate. It is assumed that the estate is not large enough to satisfy all claims, so  $E \leq \sum_{i \in N} d_i$ . We denote the class of all bankruptcy situations with player set  $N$  by  $BR^N$ .

A bankruptcy rule  $f$  is a function  $f : BR^N \rightarrow \mathbb{R}_+^N$  assigning to each bankruptcy situation  $(E, d) \in BR^N$  a payoff vector  $f(E, d) \in \mathbb{R}_+^N$ , such that  $\sum_{i \in N} f_i(E, d) = E$  and  $f(E, d) \leq d$ .

One can associate a bankruptcy game  $v_{E,d} \in TU^N$  with a bankruptcy problem  $(E, d) \in BR^N$ . The worth of a coalition  $S$  is determined by the amount of  $E$  that is not claimed by  $N \setminus S$ , so for all  $S \subset N$ ,

$$v_{E,d}(S) = \max \left\{ 0, E - \sum_{i \in N \setminus S} d_i \right\}.$$

This class of games is a proper subset of the class of compromise admissible games.

### 3 Compromise solutions based on bankruptcy

This sections introduces a new class of solutions for compromise admissible games based on bankruptcy rules. Furthermore, we take a dual approach and show that for solutions based on self-dual bankruptcy rules, the two approaches coincide.

Bankruptcy rules can be extended to solutions on the class of compromise admissible games in the following way. Let  $f : BR^N \rightarrow \mathbb{R}^N$  be a bankruptcy rule. Then the *compromise extension* of  $f$ ,  $f^*$ , is defined by

$$f^*(v) = m(v) + f(v(N) - \sum_{i \in N} m_i(v), M(v) - m(v))$$

for all  $v \in CA^N$ . Note that because  $v \in CA^N$ , the bankruptcy situation to which  $f$  is applied is well-defined. Generally, if  $f$  is a bankruptcy rule and

$f^*$  is its compromise extension, then  $f^*$  will be efficient ( $\sum_{i \in N} f_i^*(v) = v(N)$  for all  $v \in CA^N$ ). The following lemma shows that if  $f$  is homogeneous of degree 1, then  $f^*$  is relatively invariant with respect to strategic equivalence, i.e.,  $f^*(kv + a) = kf^*(v) + a$  for all  $v \in CA^N$ ,  $k > 0$  and  $a \in \mathbb{R}^N$ .

**Lemma 3.1** *If  $f(kE, kd) = k \cdot f(E, d)$  for all  $k > 0$  and all  $(E, d) \in BR^N$ , then  $f^*$  is relatively invariant with respect to strategic equivalence.*

**Proof:** Let  $v \in CA^N$ ,  $k > 0$  and  $a \in \mathbb{R}^N$ . Define  $\hat{v} = kv + a$ . Then  $\hat{v} \in CA^N$ ,  $M(\hat{v}) = kM(v) + a$  and  $m(\hat{v}) = km(v) + a$ . From this, we have

$$\begin{aligned}
f^*(\hat{v}) &= m(\hat{v}) + f(\hat{v}(N) - \sum_{i \in N} m_i(\hat{v}), M(\hat{v}) - m(\hat{v})) \\
&= km(v) + a + f(k(v(N) - \sum_{i \in N} m_i(v)), k(M(v) - m(v))) \\
&= k(m(v) + f(v(N) - \sum_{i \in N} m_i(v), M(v) - m(v))) + a \\
&= kf^*(v) + a
\end{aligned}$$

Hence,  $f^*$  is relatively invariant with respect to strategic equivalence.  $\square$

It is immediately clear that the compromise value (or  $\tau$  value) introduced by Tijs (1981) is the compromise extension of the proportional rule, since  $\tau$  is the efficient convex combination of the vectors  $M(v)$  and  $m(v)$ . Quant et al. (2003) consider the compromise extension of the Talmud rule (cf. Aumann and Maschler (1985)) for games for which the core cover coincides with the core. They prove that for this specific class of games the compromise extension of the Talmud rule equals the nucleolus. González-Díaz et al. (2003) study the compromise extension of the adjusted proportional rule (cf. Curiel et al. (1988)) and show that it coincides with the barycentre of the edges of the core cover.

Another way to extend a bankruptcy rule to an allocation rule on  $CA^N$  is to take a dual approach. Instead of first giving each player his minimum right and then dividing what is left, one could first give each player his utopia demand and take back the excess amount using  $f$ . This dual extension of a bankruptcy rule  $f$ ,  $f^\star$ , is defined by

$$f^\star(v) = M(v) - f\left(\sum_{i \in N} M_i(v) - v(N), M(v) - m(v)\right)$$

for all  $v \in CA^N$ .

The *dual* of a bankruptcy rule  $f$  (cf. Aumann and Maschler (1985)),  $\bar{f}$  is defined by

$$\bar{f}(E, d) = d - f\left(\sum_{i \in N} d_i - E, d\right)$$

and a rule is called *self-dual* if  $f = \bar{f}$ .

As is stated in the following proposition, first taking the dual of  $f$  and then extending this rule yields the same solution as taking the dual extension of  $f$ .

**Proposition 3.1** *Let  $f : BR^N \rightarrow \mathbb{R}^N$  be a bankruptcy rule and let  $v \in CA^N$ . Then  $\bar{f}^*(v) = f^\star(v)$ .*

**Proof:** Applying the definitions yields

$$\begin{aligned} \bar{f}^*(v) &= m(v) + \bar{f}\left(v(N) - \sum_{i \in N} m_i(v), M(v) - m(v)\right) \\ &= m(v) + M(v) - m(v) - \\ &\quad f\left(\sum_{i \in N} M_i(v) - \sum_{i \in N} m_i(v) - (v(N) - \sum_{i \in N} m_i(v)), M(v) - m(v)\right) \\ &= M(v) - f\left(\sum_{i \in N} M_i(v) - v(N), M(v) - m(v)\right) \\ &= f^\star(v). \end{aligned}$$

□

As a corollary, we obtain that if  $f$  is self-dual, then  $f^\star = f^*$ .

## 4 Run-to-the-bank rule

In this section we consider the compromise extension of the run-to-the-bank rule. We provide an interpretation in terms of larginals.

Let  $(E, d) \in BR^N$ . Then the run-to-the-bank rule (*RTB*) is for all  $i \in N$  defined by

$$RTB(E, d) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} r^\sigma(E, d),$$

where for  $\sigma \in \Pi(N)$ ,  $k \in \{1, \dots, |N|\}$ ,

$$r_{\sigma(k)}^\sigma(E, d) = \max \left\{ \min \{d_{\sigma(k)}, E - \sum_{r=1}^{k-1} d_{\sigma(r)}\}, 0 \right\}.$$

**Example 4.1** Let  $v \in CA^N$  with  $N = \{1, 2, 3\}$  be the game defined by

$S$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	$N$
$v(S)$	0	0	0	3	2	4	6

Then  $M(v) = (2, 4, 3)$  and  $m(v) = (0, 1, 0)$ . The larginals are given in the table below.

$\sigma$	123	132	213	231	312	321
$\ell^\sigma(v)$	(2, 4, 0)	(2, 1, 3)	(2, 4, 0)	(0, 4, 2)	(2, 1, 3)	(0, 3, 3)

The  $RTB^*$  solution equals

$$\begin{aligned} RTB^*(v) &= (0, 1, 0) + RTB(5, (2, 3, 3)) \\ &= (0, 1, 0) + \frac{1}{6}(8, 11, 11) = \left(\frac{8}{6}, \frac{17}{6}, \frac{11}{6}\right). \end{aligned}$$

Note that the  $RTB^*$  solution coincides with the average of the larginals.  $\triangleleft$

The  $RTB^*$  solution is similar to the Shapley value in the sense that it is the average of all larginals (rather than marginals)<sup>1</sup>. This is shown in the following theorem.

**Theorem 4.1** Let  $v \in CA^N$ . Then  $RTB^*(v) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} \ell^\sigma(v)$ .

**Proof:** Consider the game  $w$  defined by  $w(S) = v(S) - \sum_{i \in S} m_i(v)$  for all  $S \subset N$ . Then  $w \in CA^N$  and  $\ell^\sigma(w) = \ell^\sigma(v) - m(v)$  for all  $\sigma \in \Pi(N)$ ,  $m(w) = 0$  and  $M(w) = M(v) - m(v)$ . Next, it is readily seen that  $\ell^\sigma(w) = r^\sigma(w(N), M(w))$  for all  $\sigma \in \Pi(N)$  and hence,

$$\begin{aligned} RTB^*(w) &= RTB(w(N), M(w)) \\ &= \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} r^\sigma(w(N), M(w)) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} \ell^\sigma(w). \end{aligned}$$

As a result of Lemma 3.1,  $RTB^*$  is relatively invariant with respect to strategic equivalence and hence,

$$\begin{aligned} RTB^*(v) &= m(v) + RTB^*(w) = m(v) + \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} \ell^\sigma(w) \\ &= m(v) + \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} [\ell^\sigma(v) - m(v)] = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} \ell^\sigma(v). \end{aligned}$$

□

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<sup>1</sup>Note however that larginals do not satisfy additivity, since the minimum right vector  $m$  is not additive.



## 5 A recursive formula of $RTB^*$

In this section we give a characterisation of  $RTB^*$  by means of a recursive formula, which is based on O'Neill's consistency property.

The RTB rule is the unique bankruptcy rule satisfying the following property (cf. O'Neill (1982)):

$$f_i(E, d) = \frac{1}{|N|} \left( \min\{d_i, E\} + \sum_{j \in N \setminus \{i\}} f_i(N \setminus \{j\}, E - \min\{d_j, E\}, d_{N \setminus \{j\}}) \right)$$

for all  $(E, d) \in BR^N$  and all  $i \in N$ .<sup>2</sup>

To extend this recursive expression to our framework of compromise admissible games, we have to define a subclass which is closed with respect to “sending one player away with his claim”. For this, we need a nonnegativity condition (which is harmless as a result of relative invariance with respect to strategic equivalence) and a weak version of superadditivity. The class  $\mathcal{A}^N \subset CA^N$  consists of all TU games  $v \in CA^N$  such that for all  $S \subset N$ ,

- (i)  $v(S) \geq 0$ ,
- (ii)  $v(S) + \sum_{k \in N \setminus S} m_k(v) \leq v(N)$ .

We denote  $\mathcal{A} = \bigcup_N \mathcal{A}^N$ .

We are going to characterise  $RTB^*$  on  $\mathcal{A}$  using the following extension of O'Neill's consistency property to a solution  $f$  on  $\mathcal{A}$ : for all  $N$ , all  $v \in \mathcal{A}^N$  and all  $i \in N$  we have

$$\begin{aligned} f_i(v) &= \frac{1}{|N|} \min \left\{ M_i(v), v(N) - \sum_{j \in N \setminus \{i\}} m_j(v) \right\} + \\ &\quad \frac{1}{|N|} \sum_{j \in N \setminus \{i\}} (m_i(v) + f_i(v^j)), \end{aligned} \tag{1}$$

where the game  $v^j \in TU^{N \setminus \{j\}}$  is defined by

$$v^j(S) = \max \left\{ v(S \cup \{j\}) - \sum_{k \in S} m_k(v) - M_j(v), 0 \right\}$$

for all  $S \subset N \setminus \{j\}$ ,  $j \in N \setminus \{i\}$ .

The game  $v^j$  is again an element of  $\mathcal{A}$ , as is shown in the following lemma.

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<sup>2</sup>O'Neill calls this property consistency, which is not to be confused with various other notions of consistency.

**Lemma 5.1** *Let  $v \in \mathcal{A}^N$  and  $j \in N$ . Then  $v^j \in \mathcal{A}^{N \setminus \{j\}}$ .*

**Proof:** It is immediately clear from the definition of  $v^j(S)$  that  $v^j(S) \geq 0$  for all  $S \subset N \setminus \{j\}$ , so  $v^j$  satisfies (i). We first show that we can restrict ourselves to the case that  $v^j(N \setminus \{j\}) > 0$ .

It follows from condition (ii) applied to  $(N, v)$  that for all  $S \subset N \setminus \{j\}$  we have

$$v(S \cup \{j\}) - \sum_{i \in S} m_i(v) - M_j(v) \leq v(N) - \sum_{i \in N \setminus \{j\}} m_i(v) - M_j(v),$$

from which it easily follows that

$$v^j(S) \leq v^j(N \setminus \{j\}). \quad (2)$$

Hence, if  $v^j(N \setminus \{j\}) = 0$ , then  $v^j(S) = 0$  for all  $S \subset N \setminus \{j\}$  and  $v^j \in \mathcal{A}^{N \setminus \{j\}}$  follows trivially. So, assume that  $v^j(N \setminus \{j\}) > 0$ . It remains to prove that  $v^j$  satisfies condition (ii) and that  $v^j \in CA^{N \setminus \{j\}}$ .

**Step 1:** In order to show that  $v^j$  satisfies (ii), we calculate  $M(v^j)$  and  $m(v^j)$ . Let  $i \in N$ . If  $v^j(N \setminus \{i, j\}) = 0$ , then  $M_i(v^j) = v^j(N \setminus \{j\})$ . Otherwise,

$$\begin{aligned} M_i(v^j) &= v^j(N \setminus \{j\}) - v^j(N \setminus \{i, j\}) \\ &= v(N) - v(N \setminus \{i\}) - m_i(v) = M_i(v) - m_i(v). \end{aligned}$$

Combining the two cases, we obtain

$$M_i(v^j) = \min \{M_i(v) - m_i(v), v^j(N \setminus \{j\})\}.$$

We next show that  $m_i(v^j) = 0$ . To do so, we prove that for each  $S \subset N \setminus \{j\}$  and  $i \in S$  we have

$$\rho_i^S \leq 0, \quad (3)$$

where  $\rho_i^S = v^j(S) - \sum_{k \in S \setminus \{i\}} M_k(v^j)$ . Since  $m_i(v^j) \geq v^j(\{i\}) \geq 0$  this proves that  $m_i(v^j) = 0$ .

Let  $S \subset N \setminus \{j\}, i \in S$ . If  $v^j(S) = 0$ , then (3) follows from the fact that  $M_k(v^j) \geq 0$  for all  $k \in N \setminus \{j\}$ . Assume that  $v^j(S) > 0$ . We consider two cases.

*Case 1:*  $M_k(v^j) = M_k(v) - m_k(v)$  for all  $k \in S \setminus \{i\}$ . Then

$$\begin{aligned}\rho_i^S &= v^j(S) - \sum_{k \in S \setminus \{i\}} (M_k(v) - m_k(v)) \\ &= v(S \cup \{j\}) - \sum_{k \in S} m_k(v) - M_j(v) - \sum_{k \in S \setminus \{i\}} (M_k(v) - m_k(v)) \\ &= v(S \cup \{j\}) - \sum_{k \in S \cup \{j\} \setminus \{i\}} M_k(v) - m_i(v) \leq 0,\end{aligned}$$

where the inequality follows from the definition of  $m_i(v)$ .

*Case 2:* There exists a  $k \in S \setminus \{i\}$  with  $M_k(v^j) = v^j(N \setminus \{j\})$ . Then

$$\rho_i^S = v^j(S) - v^j(N \setminus \{j\}) - \sum_{\ell \in S \setminus \{i, k\}} M_\ell(v^j) \leq 0,$$

because of (2) and  $M_\ell(v^j) \geq 0$  for all  $\ell \in N \setminus \{j\}$ .

Hence,  $\rho_i^S \leq 0$  for all  $S \subset N \setminus \{j\}, i \in S$  and  $m(v^j) = 0$ . Condition (ii) then directly follows from (2).

**Step 2:** We next show that  $v^j \in CA^N$ . We already have  $m(v^j) = 0 \leq M(v^j)$  and  $\sum_{i \in N \setminus \{j\}} m_i(v^j) = 0 \leq v^j(N \setminus \{j\})$ . Furthermore, for  $k \in N \setminus \{j\}$  we have

$$\begin{aligned}v^j(N \setminus \{j\}) - \sum_{i \in N \setminus \{j\}} M_i(v^j) &= v^j(N \setminus \{j\}) - \sum_{i \in N \setminus \{j, k\}} M_i(v^j) - M_k(v^j) \\ &= \rho_k^{N \setminus \{j\}} - M_k(v^j) \leq 0.\end{aligned}$$

Hence,  $v^j \in CA^{N \setminus \{j\}}$ .

Because  $v^j$  also satisfy (i) and (ii), we have  $v^j \in \mathcal{A}^{N \setminus \{j\}}$ . □

The following theorem characterises  $RTB^*$  on  $\mathcal{A}$ .

**Theorem 5.1**  *$RTB^*$  is the unique solution on  $\mathcal{A}$  satisfying (1).*

**Proof:** Let  $f$  be a solution on  $\mathcal{A}$  satisfying (1). Then this uniquely determines the outcome of  $f$  for all one-player games and, by induction, for all games in  $\mathcal{A}$ . Therefore, there can only be one rule that satisfies (1) on  $\mathcal{A}$ . Hence, it suffices to show that  $RTB^*$  satisfies (1) on  $\mathcal{A}$ .

Let  $v \in \mathcal{A}^N$ . From (O'Neill) consistency of  $RTB$  it follows that for all  $i \in N$  we have

$$\begin{aligned} RTB_i^*(v) &= m_i(v) + RTB_i(N, v(N) - \sum_{j \in N} m_j(v), M(v) - m(v)) \\ &= m_i(v) + \frac{1}{|N|} \left[ \min\{M_i(v) - m_i(v), v(N) - \sum_{j \in N} m_j(v)\} + \right. \\ &\quad \left. \sum_{j \in N \setminus \{i\}} RTB_i(N \setminus \{j\}, E_{-j}, d_{-j}) \right] \end{aligned}$$

with  $E_{-j} = v(N) - \sum_{j \in N} m_j(v) - \min\{M_j(v) - m_j(v), v(N) - \sum_{j \in N} m_j(v)\}$  and  $d_{-j} = M_{-j} - m_{-j} = (M_k(v) - m_k(v))_{k \in N \setminus \{j\}}$ . Note that by construction  $E_{-j} = v^j(N \setminus \{j\})$ . Then,

$$\begin{aligned} RTB_i^*(v) &= \frac{1}{|N|} \min\{M_i(v), v(N) - \sum_{j \in N \setminus \{i\}} m_j(v)\} + \\ &\quad \frac{1}{|N|} \sum_{j \in N \setminus \{i\}} \left( m_i(v) + RTB_i(N \setminus \{j\}, v^j(N \setminus \{j\}), M_{-j} - m_{-j}) \right). \end{aligned}$$

Since  $(M_{-j} - m_{-j})_k \geq \min\{v^j(N \setminus \{j\}), M_{-j} - m_{-j}\} = M_k(v^j)$  for all  $k \in N \setminus \{j\}$ , the truncation property (ie, for all  $(E, d) \in BR^N$ ,  $RTB(E, d) = RTB(E, d')$  with  $d'_i = \min\{E, d_i\}$  for all  $i \in N$ ) gives

$$\begin{aligned} RTB_i^*(v) &= \frac{1}{|N|} \min\{M_i(v), v(N) - \sum_{j \in N \setminus \{i\}} m_j(v)\} + \\ &\quad \frac{1}{|N|} \sum_{j \in N \setminus \{i\}} \left( m_i(v) + RTB_i(N \setminus \{j\}, v^j(N \setminus \{j\}), M(v^j)) \right) \\ &= \frac{1}{|N|} \min\{M_i(v), v(N) - \sum_{j \in N \setminus \{i\}} m_j(v)\} + \\ &\quad \frac{1}{|N|} \sum_{j \in N \setminus \{i\}} (m_i(v) + RTB_i^*(v^j)), \end{aligned}$$

where the last equality is true, because  $m(v^j) = 0$  for all  $j \in N$ . Hence,  $RTB^*$  satisfies (1).  $\square$

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